

# Integral Equation for the Ahlfors Map on Multiply Connected Regions

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## Article history

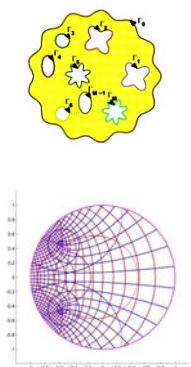
Received: 2 July 2014

Received in revised form:

2 December 2014

Accepted: 1 February 2015

## Graphical abstract



## Abstract

This paper presents a new boundary integral equation with the adjoint Neumann kernel associated with  $\theta'$ , where  $\theta$  is the boundary correspondence function of Ahlfors map of a bounded multiply connected region onto a unit disk. The proposed boundary integral equation is constructed from a boundary relationship satisfied by the Ahlfors map of a multiply connected region. The integral equation is solved numerically for  $\theta'$ , using combination of Nystrom method, GMRES method, and fast multiple method. From the computed values of  $\theta'$ , we solve for the boundary correspondence function  $\theta$  which then gives the Ahlfors map. The numerical examples presented here prove the effectiveness of the proposed method.

**Keywords:** Ahlfors map, adjoint Neumann kernel, Generalized Neumann kernel, GMRES, fast multipole method.

## Abstrak

Kertas kerja ini memberikan satu persamaan kamiran sempadan baru dengan inti Neumann adjoin berkaitan dengan  $\theta'$ , dengan  $\theta$  ialah fungsi kesepadanan sempadan terhadap pemetaan Ahlfors dari rantau terkait berganda terbatas ke cakera unit. Persamaan kamiran sempadan ini dibina dari hubungan sempadan yang ditepati oleh pemetaan Ahlfors map tersebut. Persamaan kamiran tersebut diselesaikan secara berangka untuk  $\theta'$  menggunakan kombinasi kaedah on of Nystrom, kaedah GMRES, dan kaedah multikutub pantas. Dari nilai  $\theta'$  yang telah dihitung, kita selesaikan fungsi kesepadanan sempadan  $\theta$  lalu menghasilkan pemetaan Ahlfors. Contoh-contoh berangka yang dibentangkan menunjukkan keberkesanan kaedah yang dibincangkan.

**Katakunci :** Pemetaan Ahlfors, inti Neumann adjoin, Inti Neumann teritlak, GMRES, kaedah multikutub pantas.

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## 1.0 INTRODUCTION

Conformal mapping is a useful tool for solving various problems of science and engineering such as fluid flow, electrostatics, heat conduction, mechanics, aerodynamics, and image processing. The conformal mapping from a multiply connected region onto the unit disk is known as the Ahlfors map. If the region is simply connected then the Ahlfors map reduces to the Riemann map. Many of the geometrical features of a Riemann mapping function are shared with Ahlfors map. The Riemann mapping function can be regarded as a solution of the following extremal problem:

For a simply connected region  $\Omega$  and canonical region  $D$  in the complex plane  $\mathbb{C}$  and fixed  $a$  in  $\Omega$ , construct an extremal analytic map

$$F: \Omega \rightarrow D \text{ with } F'(a) > 0.$$

The Riemann map is the solution of this problem. It is unique conformal, one-to-one and onto map with  $F(a) = 0$ .

For a multiply connected region  $\Omega$  of connectivity  $n > 1$ , the answer to the same extremal problem above becomes the Ahlfors map. It is unique analytic map

$$f: \Omega \rightarrow D$$

that is onto, if  $f'(a) > 0$  and  $f(a) = 0$ . However it has  $2n - 2$  branch points in the interior and is no longer one-to-one there. In fact it maps  $\Omega$  onto  $D$  in an  $n$ -to-one fashion, and maps each boundary curve one-to-one onto the unit circle (see [1] and [2]). Therefore the Ahlfors map can be regarded as the Riemann mapping function in the multiply connected region.

Conformal mapping of multiply connected regions can be computed efficiently using the integral equation method. The integral equation method has been used by many authors to compute the one-to-one conformal mapping from multiply connected regions onto some standard canonical regions [3-16]. In [10] the authors presented a fast multipole method, which is fast and accurate method for numerical conformal mapping of bounded and unbounded multiply connected regions with high connectivity and highly complex geometry. The authors used the Matlab function `zfm2dpart` in the MATLAB toolbox FMMLIB2D developed by Greengard and Gimbutas [17] as a

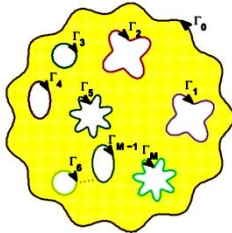
vector product function for the coefficient matrix of the linear system.

Some integral equations for Ahlfors map have been given in [2, 18–21]. In [3] Kerzman and Stein have derived a uniquely solvable boundary integral equation for computing the Szegő kernel of a bounded region and this method has been generalized in [18] to compute Ahlfors map of bounded multiply connected regions without relying on the zeros of Ahlfors map. In [19, 20] the integral equations for Ahlfors map of doubly connected regions requires knowledge of zeros of Ahlfors map, which are unknown in general.

In this paper, we extend the approach of Sangawi [12, 14, 15] to construct an integral equation for the Ahlfors map of multiply connected region onto a unit disk. The plan of this paper is as follows: After presenting some auxiliary materials in Section 2, we shall derive in Section 3, a boundary integral equation satisfied by  $\theta'$ , where  $\theta$  is the boundary correspondence function of Ahlfors map of bounded multiply connected regions onto a disk. From the computed values of  $\theta'$ , we then determine the Ahlfors map. In Section 4, we present some examples to illustrate our boundary integral equation method. The numerical examples are restricted to annulus region for which the exact Ahlfors map is known which allows for the numerical comparison between our proposed method with the exact Ahlfors map. Finally, Section 6 presents a short conclusion.

## 2.0 AUXILIARY MATERIAL

Let  $\Omega$  be a bounded multiply connected region of connectivity  $M + 1$ . The boundary  $\Gamma$  consists of  $M + 1$  smooth Jordan curves  $\Gamma_0, \Gamma_1, \dots, \Gamma_M$  such that  $\Gamma_1, \dots, \Gamma_M$  lie in the interior of  $\Gamma_0$ , where the outer curve  $\Gamma_0$  has counterclockwise orientation and inner curves  $\Gamma_1, \dots, \Gamma_M$  have clockwise orientation. The positive direction of the contour  $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \dots \cup \Gamma_M$  is usually that for which  $\Omega$  is on the left as one traces the boundary as shown in Figure 1.



**Figure 1** A bounded multiply connected region of connectivity  $M+1$

The curves  $\Gamma_j$  are parameterized by  $2\pi$ -periodic twice continuously differentiable complex-valued functions  $z_j(t)$  with non-vanishing first derivatives

$$z'_j(t) = dz_j(t)/dt \neq 0, \quad t \in J_j = [0, 2\pi], \\ j = 0, 1, \dots, M.$$

The total parameter domain  $J$  is defined as the disjoint union of  $M + 1$  intervals  $J_0, J_1, \dots, J_M$ . The notation

$$z(t) = z_j(t), \quad t \in J_j, \quad j = 0, 1, \dots, M. \quad (2.1)$$

is interpreted as follows [20]: For a given  $\tilde{t} \in [0, 2\pi]$ , to evaluate the value of  $z(t)$  at  $\tilde{t}$ , we should know in advance the interval  $J_j$  to which  $\tilde{t}$  belongs, i.e. we should know the boundary  $\Gamma_j$  contains  $z(\tilde{t})$ , then we compute  $z(\tilde{t}) = z_j(\tilde{t})$ .

The generalized Neumann kernel formed with a complex continuously differentiable  $2\pi$ -periodic function  $A(t)$  for all  $t \in J$ , is defined by [22, 23]

$$\hat{N}(t, s) = \frac{1}{\pi} \operatorname{Im} \left( \frac{A(t)}{A(s)} \frac{z'(s)}{z(s) - z(t)} \right). \quad (2.2)$$

For  $A(t) = 1$ ,  $\hat{N}(t, s)$  reduces to the well-known classical Neumann kernel  $N(t, s)$ .

The kernel is continuous which takes on the diagonal the values with

$$\hat{N}(t, t) = \frac{1}{\pi} \left( \frac{1}{2} \operatorname{Im} \frac{z''(t)}{z'(t)} - \operatorname{Im} \frac{A'(t)}{A(t)} \right).$$

It follows from the representation (2.2) that the adjoint kernel

$$N^*(s, t) = \hat{N}(t, s) = \frac{1}{\pi} \operatorname{Im} \left( \frac{A(t)}{A(s)} \frac{z'(s)}{z(s) - z(t)} \right),$$

can be represented as

$$N^*(s, t) = -\frac{1}{\pi} \operatorname{Im} \left( \frac{\tilde{A}(s)}{\tilde{A}(t)} \frac{z'(t)}{z(t) - z(s)} \right),$$

with the adjoint function  $\tilde{A}(t)$  is given by

$$\tilde{A}(t) = \frac{z'(t)}{A(t)}. \quad (2.3)$$

The generalized Neumann kernel  $\tilde{N}(s, t)$  formed with  $\tilde{A}(t)$  is given by

$$\tilde{N}(t, s) = \frac{1}{\pi} \operatorname{Im} \left( \frac{\tilde{A}(t)}{\tilde{A}(s)} \frac{z'(s)}{z(s) - z(t)} \right),$$

which implies

$$\tilde{N}(s, t) = -N^*(s, t).$$

It is also known that  $\lambda = 1$  is an eigenvalue of the kernel  $N$  with multiplicity 1 and  $\lambda = -1$  is an eigenvalue of the kernel  $N$  with multiplicity  $M$  [23]. The eigenfunctions of  $N$  corresponding to the eigenvalue  $\lambda = -1$  are  $\{\chi^{[1]}, \chi^{[2]}, \dots, \chi^{[M]}\}$ , where

$$\chi^{[j]}(\zeta) = \begin{cases} 1, & \zeta \in \Gamma_j, \\ 0, & \text{otherwise, } j = 1, \dots, M. \end{cases}$$

Let  $H^*$  be the space of all real Hölder continuous  $2\pi$ -periodic functions  $\omega(t)$  of the parameter  $t$  on  $J_j$  for  $j = 0, 1, \dots, M$ , i.e.

$$\omega(t) = \omega_j(t), \quad t \in J_j, \quad j = 0, 1, \dots, M.$$

We define the space  $S$  by

$$S = \operatorname{span}\{\chi^{[0]}, \chi^{[1]}, \dots, \chi^{[M]}\},$$

and the integral operators  $\mathbf{J}$  by [13, 15, 24]

$$\mathbf{J}v = \frac{1}{2\pi} \sum_{j=0}^M \chi^{[j]}(t) (\chi^{[j]}, v),$$

with the inner product

$$(\chi^{[j]}, v) = \int_J \chi^{[j]}(s) v(s) ds.$$

We also define the Fredholm integral operator  $N^*$  by

$$\mathbf{N}^* \psi(t) = \int_J N^*(t, s) \psi(s) ds, \quad t \in J. \quad (2.4)$$

The following theorem which can be proved by using the approach as in Theorem 5 in [24] will be useful in the next section.

### 2.1 Theorem 2.1

Suppose the function  $\gamma, \mu \in H^*$  and  $h, v \in S$  such that

$$A\hat{g}(z) = \gamma + h + i(\mu + v) \quad (2.5)$$

are the boundary values of an analytic function  $\hat{g}(z)$  in  $\Omega$ . Then the functions  $h = (h_0, h_1, \dots, h_M)$  and  $(v_0, v_1, \dots, v_M)$  are given by

$$h_j = (\gamma, \phi^{[j]}) = \frac{1}{2\pi} \int_{\Gamma} \gamma(t) \phi^{[j]}(t) dt, \quad (2.6)$$

$$v_j = (\mu, \phi^{[j]}) = \frac{1}{2\pi} \int_{\Gamma} \mu(t) \phi^{[j]}(t) dt. \quad (2.7)$$

where  $\varphi^{[j]}$  are solutions of the following integral equations

$$(\mathbf{I} + \mathbf{N}^* + \mathbf{J})\varphi^{[j]} = -\chi^{[j]}, \quad j = 0, 1, \dots, M. \quad (2.8)$$

A complex-valued function  $P(z)$  is said to satisfy the interior relationship if  $P(z)$  is analytic in  $\Omega$  and satisfies the non-homogeneous boundary relationship

$$P(z) = \frac{b(z)\overline{T(z)}}{G(z)} \overline{P(z) + H(z)}, \quad (2.9)$$

where  $G(z)$  is analytic in  $\Omega$ , Hölder continuous on  $\Gamma$ , and  $G(z) \neq 0$  on  $\Gamma$ . The boundary relationship (2.9) also has the following equivalent form:

$$G(z) = \overline{b(z)T(z)} \frac{P(z)^2}{|P(z)|^2} + \frac{G(z)H(z)}{P(z)}. \quad (2.10)$$

The following theorem gives an integral equation for an analytic function satisfying the interior non-homogeneous boundary relationship (2.9) or (2.10).

## 2.2 Theorem 2.2 [14]

If the function  $P(z)$  satisfies the interior relationship (2.9) or (2.10), then

$$T(z)P(z) + \int_{\Gamma} K(z, w)T(w)P(w) |dw| + b(z) \left[ \sum_{a_j \text{ inside } \Gamma} \text{Res}_{w=a_j} \frac{P(w)}{(w-z)G(w)} \right] = -\overline{L_R^-(z)}, \quad (2.11)$$

where

$$K(z, w) = \frac{1}{2\pi i} \left[ \frac{T(z)}{z-w} - \frac{b(z)\overline{T(w)}}{b(w)(\overline{z}-\overline{w})} \right], \quad (2.12)$$

and

$$L_R^-(z) = \frac{-1}{2} \frac{H(z)}{T(z)} + \text{PV} \frac{1}{2\pi i} \int_{\Gamma} \frac{\overline{b(z)}H(w)}{\overline{b(w)}(w-z)T(w)} dw. \quad (2.13)$$

The symbol “—” in the superscript denotes the complex conjugate and the sum is over all those zeros that lie inside  $\Omega$ . If  $G$  has no zeros in  $\Omega$ , then the term containing the residue will not appear.

## 3.0 INTEGRAL EQUATION METHOD FOR COMPUTING $\theta'$

Let  $f(z)$  be the Ahlfors function which maps  $\Omega$  conformally onto a unit disc. The mapping function  $f$  is determined up to a factor of modulus 1. The function  $f$  could be made unique by imposing the condition

$$f(a_j) = 0, \quad f'(a_0) > 0, \quad j = 0, 1, \dots, M.$$

where  $a_j \in \Omega$ ,  $j = 0, 1, \dots, M$  are the zeros of the Ahlfors map. The boundary values of  $f$  can be represented as

$$f(z_j(t)) = e^{i\theta_j(t)}, \quad \Gamma_j : z = z_j(t), \quad 0 \leq t \leq \beta_j, \quad (3.1)$$

where  $\theta_j(t)$ ,  $j = 0, 1, \dots, M$  are the boundary correspondence functions of  $\Gamma_j$ . The unit tangent to  $\Gamma$  at  $z(t)$  is denoted by

$$T(z(t)) = \frac{z'(t)}{|z'(t)|}.$$

We have from (3.1) that

$$f'(z_j(t))z'_j(t) = e^{i\theta_j(t)} i\theta'_j(t) = f(z_j(t))i\theta'_j(t). \quad (3.2)$$

Taking modulus on both sides of (3.2), we get

$$|f'(z_j(t))||z'_j(t)| = |\theta'_j(t)|. \quad (3.3)$$

Dividing (3.2) by (3.3)

$$\frac{f'(z_j(t))z'_j(t)}{|f'(z_j(t))||z'_j(t)|} = \frac{f(z_j(t))i\theta'_j(t)}{|\theta'_j(t)|}.$$

Equivalently, we have

$$f(z_j(t)) = \frac{1}{i} T(z_j(t)) \frac{|\theta'_j(t)|}{\theta'_j(t)} \frac{f'(z_j(t))}{|f'(z_j(t))|}, \quad z_j \in \Gamma_j. \quad (3.4)$$

By the angle preserving property of conformal map, the image of  $\Gamma_0$  remains in counter-clockwise orientation so  $\theta'_0(t) > 0$ , while the images of inner boundaries  $\Gamma_j$  in clockwise orientation therefore  $\theta'_j(t) < 0$ , for  $j = 1, 2, \dots, M$ . Thus

$$\frac{|\theta'_j(t)|}{\theta'_j(t)} = \text{sign}(\theta'_j(t)) = \begin{cases} +1, & j = 0, \\ -1, & j = 1, \dots, M. \end{cases}$$

The boundary relationship (3.4) can be written briefly as

$$f(z) = \text{sign}(\theta'_j(t)) \frac{1}{i} T(z) \frac{f'(z)}{|f'(z)|}, \quad z \in \Gamma. \quad (3.5)$$

Since the Ahlfors map can be written as

$$f(z) = \prod_{j=0}^M (z - a_j) g(z), \quad (3.6)$$

where  $g(z)$  is analytic in  $\Omega$  and  $g(z) \neq 0$  in  $\Omega$ , we have

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + \sum_{j=0}^M \frac{1}{z - a_j}, \quad z \in \Gamma, \quad (3.7)$$

so that

$$D(z) = \frac{f'(z)}{f(z)} - \sum_{j=0}^M \frac{1}{z - a_j} = \frac{g'(z)}{g(z)}. \quad (3.8)$$

is analytic in  $\Omega$ . Squaring both sides of (3.5), gives

$$f(z)^2 = -T(z)^2 \frac{f'(z)^2}{|f'(z)|^2}, \quad (3.9)$$

which can be written as

$$f(z)f'(z) = -T(z)^2 \frac{f'(z)f'(z)}{f'(z)f'(z)}. \quad (3.10)$$

Using the fact that

$$f(z) = \frac{1}{f(z)}.$$

Eq. (3.10) becomes

$$\frac{f(z)}{f'(z)} = -T(z)^2 \frac{f'(z)}{f'(z)},$$

or

$$\frac{\overline{f'(z)}}{f(z)} = -T(z)^2 \frac{f'(z)}{f(z)}.$$

Taking conjugate to both sides, we get

$$\frac{f'(z)}{f(z)} = -\overline{T(z)}^2 \left( \frac{f'(z)}{f(z)} \right). \quad (3.11)$$

From (3.8), we have

$$\frac{f'(z)}{f(z)} = D(z) + \sum_{j=0}^M \frac{1}{z - a_j}. \quad (3.12)$$

Substituting (3.12) into (3.11), we get

$$D(z) + \sum_{j=0}^M \frac{1}{z - a_j} = -\overline{T(z)}^2 D(z) - \sum_{j=0}^M \frac{\overline{T(z)}^2}{z - a_j}, \quad (3.13)$$

or

$$D(z) = -\overline{T(z(t))}^2 \overline{D(z)} - \sum_{j=0}^M \frac{\overline{T(z)}^2}{z - a_j} - \sum_{j=0}^M \frac{1}{z - a_j}. \quad (3.14)$$

Comparing (3.14) with (2.9), we get

$$P(z) = D(z), \quad b(z) = \overline{T(z)}, \quad G(z) = 1, \quad H(z) = -\sum_{j=0}^M \frac{T(z)^2}{z - a_j} - \sum_{j=0}^M \frac{1}{z - a_j}. \quad (3.15)$$

Substituting these assignments into Theorem 2.2, we obtain

$$T(z)D(z) + \int_{\Gamma} K(z, w)T(w)D(w) |dw| = -\overline{L_R^-(z)}, \quad (3.16)$$

where

$$K(z, w) = \frac{1}{2\pi i} \left[ \frac{T(z)}{z - w} - \frac{\overline{T(z)}}{\overline{z - w}} \right] = \frac{1}{\pi} \operatorname{Im} \left( \frac{T(z)}{z - w} \right) = N(z, w) \quad (3.17)$$

is the classical Neumann kernel  $N(z, w)$ , and

$$L_R^-(z) = \frac{-1}{2T(z)} \left[ -\sum_{j=0}^M \frac{T(z)^2}{z - a_j} - \sum_{j=0}^M \frac{1}{z - a_j} \right] + PV \frac{1}{2\pi i} \int_{\Gamma} \frac{T(z)}{(w - z)T(w)^2} \left[ -\sum_{j=0}^M \frac{T(w)^2}{w - a_j} - \sum_{j=0}^M \frac{1}{w - a_j} \right] dw. \quad (3.18)$$

Substituting (3.8) into (3.16), the left-hand side of (3.16) becomes

$$\begin{aligned} LHS &= T(z) \left( \frac{f'(z)}{f(z)} - \sum_{j=0}^M \frac{1}{z - a_j} \right) + \int_{\Gamma} N(z, w)T(w) \left( \frac{f'(w)}{f(w)} - \sum_{j=0}^M \frac{1}{w - a_j} \right) |dw| \\ &= T(z) \frac{f'(z)}{f(z)} - \sum_{j=0}^M \frac{T(z)}{z - a_j} + \int_{\Gamma} N(z, w)T(w) \frac{f'(w)}{f(w)} |dw| \\ &\quad - \int_{\Gamma} N(z, w)T(w) \left( \sum_{j=0}^M \frac{1}{w - a_j} \right) |dw|. \end{aligned}$$

Using the definition of  $N(z, w)$  from (3.17) in the last integral

$$\begin{aligned} LHS &= T(z) \frac{f'(z)}{f(z)} - \sum_{j=0}^M \frac{T(z)}{z - a_j} + \int_{\Gamma} N(z, w)T(w) \frac{f'(w)}{f(w)} |dw| \\ &\quad - PV \frac{1}{2\pi} \int_{\Gamma} \left[ \frac{T(z)}{z - w} - \frac{\overline{T(z)}}{\overline{z - w}} \right] T(w) \left( \sum_{j=0}^M \frac{1}{w - a_j} \right) |dw| \\ &= T(z) \frac{f'(z)}{f(z)} - \sum_{j=0}^M \frac{T(z)}{z - a_j} + \int_{\Gamma} N(z, w)T(w) \frac{f'(w)}{f(w)} |dw| \\ &\quad + \sum_{j=0}^M T(z) \left( PV \frac{1}{2\pi i} \int_{\Gamma} \frac{T(w) |dw|}{(w - z)(w - a_j)} \right) + \sum_{j=0}^M \left( PV \frac{1}{2\pi i} \int_{\Gamma} \frac{\overline{T(z)} \overline{T(w)} |dw|}{(\overline{z - w})(\overline{w - a_j})} \right). \end{aligned}$$

Using the facts that [13]

$$PV \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{(w - z)(w - a_j)} dw = -\frac{1}{2(z - a_j)},$$

$dw = T(w)|dw|$ , and after some simplifications, we have

$$\begin{aligned} LHS &= T(z) \frac{f'(z)}{f(z)} + \int_{\Gamma} N(z, w)T(w) \frac{f'(w)}{f(w)} |dw| - \frac{3}{2} \sum_{j=0}^M \frac{T(z)}{z - a_j} \\ &\quad - \sum_{j=0}^M \left( PV \frac{1}{2\pi i} \int_{\Gamma} \frac{\overline{T(z)} dw}{(\overline{w - z})(\overline{w - a_j})} \right). \end{aligned} \quad (3.19)$$

Using the similar approach, (3.18) becomes

$$L_R(z) = \sum_{j=0}^M \frac{T(z)}{z - a_j} + \frac{1}{2} \sum_{j=0}^M \frac{\overline{T(z)}}{z - a_j} - \sum_{j=0}^M \left( PV \frac{1}{2\pi i} \int_{\Gamma} \frac{T(z) |dw|}{(w - z)(\overline{w - a_j})} \right). \quad (3.20)$$

Substituting the results from (3.19) and (3.20) into (3.16), we get

$$T(z) \frac{f'(z)}{f(z)} + \int_{\Gamma} N(z, w) \frac{f'(w)}{f(w)} T(w) |dw| = 2i \operatorname{Im} \left[ \sum_{j=0}^M \frac{T(z)}{z(t) - a_j} \right], \quad (3.21)$$

where

$$N(z, w) = \begin{cases} \frac{1}{\pi} \operatorname{Im} \left( \frac{T(z)}{z - w} \right), & z \neq w \in \Gamma, \\ \frac{1}{2\pi |z'(t)|} \operatorname{Im} \left( \frac{z''(t)}{z'(t)} \right), & z = w \in \Gamma. \end{cases} \quad (3.22)$$

In integral equation (3.21), letting  $z = z(t)$ ,  $w = z(s)$  and multiplying both sides by  $|z'(t)|$ , gives

$$\frac{f'(z(t))}{f(z(t))} z'(t) + \int_{\Gamma} N(z(t), z(s)) \frac{f'(z(s))}{f(z(s))} z'(s) ds = 2i \operatorname{Im} \left[ \sum_{j=0}^M \frac{z'(t)}{z(t) - a_j} \right]. \quad (3.23)$$

Using the fact that

$$\frac{f'(z(t))z'(t)}{f(z(t))} = i\theta'(t), \quad (3.24)$$

(3.23) becomes

$$\theta'(t) + \int_{\Gamma} N(t, s)\theta'(s) ds = 2 \operatorname{Im} \left[ \sum_{j=0}^M \frac{z'(t)}{z(t) - a_j} \right]. \quad (3.25)$$

Since  $N^*(s, t) = N(t, s)$ , the integral equation (3.25) in the operator form is

$$(\mathbf{I} + N^*)\theta' = \phi(t), \quad j = 0, 1, \dots, M. \quad (3.26)$$

where

$$\phi(t) = 2 \operatorname{Im} \left[ \sum_{j=0}^M \frac{z'(t)}{z(t) - a_j} \right]. \quad (3.27)$$

By Theorem 12 in [23],  $\lambda = -1$  is an eigenvalue of  $N^*$  with multiplicity  $M$ , therefore the integral equation (3.26) is not solvable. To overcome this problem, we note that image of the curve  $\Gamma_0$  is counterclockwise oriented, we have  $\theta_0(2\pi) - \theta_0(0) = 2\pi$ , and the image of the curves  $\Gamma_j$ ,  $j = 1, 2, \dots, M$  is clockwise oriented with

$$\theta_j(2\pi) - \theta_j(0) = -2\pi,$$

So that we have

$$\frac{1}{2\pi i} \int_{\Gamma_j} \frac{f'(w)T(w)}{f(w)} |dw| = \begin{cases} 1, & j = 0, \\ -1, & j = 1, \dots, M. \end{cases} \quad (3.28)$$

As  $\mathbf{J}$  is the disjoint union of  $M + 1$  intervals  $J_0, J_1, \dots, J_M$ , so we can write vector form of  $\mathbf{J}$  as

$$\mathbf{J}\theta' = (J\theta'_0, J\theta'_1, \dots, J\theta'_M).$$

from (3.24) and (3.28) and using the definition of  $\mathbf{J}$ , we have

$$\mathbf{J}\theta' = \left( \chi^{[0]}(t_0) \frac{1}{2\pi} \int_{J_0} \theta'(s) ds, \chi^{[1]}(t_1) \frac{1}{2\pi} \int_{J_1} \theta'(s) ds, \dots, \chi^{[M]}(t_M) \frac{1}{2\pi} \int_{J_M} \theta'(s) ds \right).$$

Thus

$$\mathbf{J}\theta' = \left( \frac{1}{2\pi} \int_{J_0} \theta'(s) ds, \frac{1}{2\pi} \int_{J_1} \theta'(s) ds, \dots, \frac{1}{2\pi} \int_{J_M} \theta'(s) ds \right).$$

which implies that

$$\mathbf{J}\theta' = \psi, \quad (3.30)$$

where

$$\psi = (1, -1, -1, \dots, -1). \quad (3.31)$$

By adding (3.26) and (3.30), we get

$$(\mathbf{I} + \mathbf{N}^* + \mathbf{J})\theta' = \phi + \psi. \quad (3.32)$$

Assuming the zeros of  $f(z)$  are known, we can solve the integral equation (3.32) for  $\theta'_j(t)$ . The boundary correspondence functions  $\theta_j(t)$  for  $j = 0, 1, \dots, M$  can be calculated from  $\theta'_j(t)$  by [15]

$$\theta_j(t) = \int_{J_j} \theta'_j(t) dt + v_j = \rho_j(t) + v_j, \quad t \in J_j, \quad (3.33)$$

where  $v_j$  are undetermined real constants and the real functions  $\rho_j(t)$  are defined by

$$\rho_j(t) = \int_{J_j} \theta'_j(t) dt, \quad t \in J_j. \quad (3.34)$$

The functions  $\theta_j(t)$  being  $2\pi$ -periodic can be represented by a Fourier series

$$\theta_j(t) = a_0^{[j]} + \sum_{k=1}^{\infty} a_k^{[j]} \cos kt + \sum_{k=1}^{\infty} b_k^{[j]} \sin kt, \quad t \in J_j. \quad (3.35)$$

Hence the functions  $\rho_j(t)$  can be calculated by the Fourier series representation as

$$\rho_j(t) = a_0^{[j]}(t) + \sum_{k=1}^{\infty} \frac{a_k^{[j]}}{k} \sin kt - \sum_{k=1}^{\infty} \frac{b_k^{[j]}}{k} \cos kt, \quad t \in J_j. \quad (3.36)$$

It now remains to determine  $v_j$ ,  $j = 0, 1, \dots, M$  in (3.33). Now as from Nasser [20], the Ahlfors map  $f$  can also be written as

$$f(z) = c(z - a_0) \prod_{j=1}^M \frac{a_0 - z_j}{a_0 - a_j} \prod_{j=1}^M \frac{z - a_j}{z - z_j} e^{(z - a_0)g(z)}. \quad (3.37)$$

Taking log on both sides of (3.35), we get

$$\begin{aligned} \log f(z) &= \log(c) + \log(z - a_0) + \sum_{j=1}^M \log\left(\frac{a_0 - z_j}{a_0 - a_j}\right) \\ &\quad + \sum_{j=1}^M \log\left(\frac{z - a_j}{z - z_j}\right) + (z - a_0)g(z). \end{aligned} \quad (3.38)$$

Applying (3.1) to the term  $\log f(z)$  and using (3.33), Eq. (3.38) becomes

$$\begin{aligned} (z - a_0)g(z) &= i\theta(t) - \log(c) - \log(z - a_0) - \sum_{j=1}^M \log\left(\frac{a_0 - z_j}{a_0 - a_j}\right) - \sum_{j=1}^M \log\left(\frac{z - a_j}{z - z_j}\right) \\ &= \operatorname{Re}\left(-\log(z - a_0) - \sum_{j=1}^M \log\left(\frac{z - a_j}{z - z_j}\right)\right) \\ &\quad + \operatorname{Re}\left(-\log(c) - \sum_{j=1}^M \log\left(\frac{a_0 - z_j}{a_0 - a_j}\right)\right) \\ &\quad + i\left[\rho + v + \operatorname{Im}\left(-\log(z - a_0) - \sum_{j=1}^M \log\left(\frac{z - a_j}{z - z_j}\right)\right)\right] \\ &\quad + \operatorname{Im}\left(-\log(c) - \sum_{j=1}^M \log\left(\frac{a_0 - z_j}{a_0 - a_j}\right)\right). \end{aligned}$$

This result has the form

$$(z - a_0)g(z) = \gamma(t) + h(t) + i(\rho + \tilde{v} + \mu(t)), \quad (3.39)$$

where

$$\gamma(t) = \operatorname{Re}\left(-\log(z - a_0) - \sum_{j=1}^M \log\left(\frac{z - a_j}{z - z_j}\right)\right), \quad (3.40a)$$

$$h(t) = \operatorname{Re}\left(-\log(c) - \sum_{j=1}^M \log\left(\frac{a_0 - z_j}{a_0 - a_j}\right)\right), \quad (3.40b)$$

$$\tilde{v} = v + \operatorname{Im}\left(-\log(c) - \sum_{j=1}^M \log\left(\frac{a_0 - z_j}{a_0 - a_j}\right)\right), \quad (3.40c)$$

$$\mu(t) = \operatorname{Im}\left(-\log(z - a_0) - \sum_{j=1}^M \log\left(\frac{z - a_j}{z - z_j}\right)\right). \quad (3.40d)$$

We now apply Theorem 2.1 to (3.39). Solving the integral equation (2.8) for  $\phi^{[j]}$ ,  $j = 0, 1, \dots, M$ , gives  $h_j$  through (2.6), and  $c$  from (3.40b). By (3.36) we get the value of  $\rho$  and through (2.7) we get the values of  $\tilde{v}_j$  which in turn give  $v_j$  from (3.40c), and finally  $\theta_j$  from (3.33). The approximate boundary values of  $f(z)$  are given by

$$f(z_j(t)) = e^{i\theta_j(t)} = e^{i(\rho_j(t) + v_j(t))}, \quad j = 0, 1, \dots, M. \quad (3.41)$$

Then the interior values of the function  $f(z)$  are calculated by the Cauchy integral formula

$$f(z) = \int_{\Gamma} \frac{f(w)}{w - z} dw, \quad z \in \Omega. \quad (3.42)$$

#### 4.0 NUMERICAL EXAMPLES

For solving the integral equation (3.32) numerically, the reliable procedure is by using the Nyström method with the trapezoidal rule with  $n$  equidistant nodes in each interval  $J_j$ ,  $j = 0, 1, \dots, M$  [6–10]. In [10] an iterative method GMRES have been applied for solving the linear systems powered by the fast multipole method (FMM). We use the similar approach as [10] to solve our linear system. For evaluating the Cauchy integral formula (3.42) numerically, we use the equivalent form

$$f(z) = \frac{\int_{\Gamma} \frac{f(w)}{w - z} dw}{\int_{\Gamma} \frac{1}{w - z} dw}, \quad z \in \Omega, \quad (4.1)$$

which also works very well for  $z \in \Omega$  near the boundary  $\Gamma$ . When the trapezoidal rule is applied to the integrals in (4.1), the term in the denominator compensates the error in the numerator (see [15]). In [2], Tegtmeier and Thomas computed the Ahlfors map using Szegő and Garabedian kernels for the annulus region, where the authors have used series representations of both Szegő kernel and Garabedian kernel. With these representations, they found the two zeros  $a_0$  and  $a_1 = \frac{r}{a_0}$  for Ahlfors map, where  $r$  is the radius of the inner circle. They have also considered the symmetry case when the zeros are  $a_0 = \sqrt{r}$  and  $a_1 = -\sqrt{r}$ . Here we shall use these values of zeros of Ahlfors map and consider a numerical example in the annulus  $r < |z| < 1$ . These examples

have also been considered in [20] where Ahlfors map was computed using a boundary integral equation related to a Riemann-Hilbert problem. Orthogonal grids over the original region  $\Omega$  are shown in Figure 2(a) and Figure 3(a) for different values of  $r$ . The images of the region  $\Omega$  for non-symmetrical are shown in Figure 2(b) and Figure 3(b) and symmetric cases in Figure 2(c) and Figure 3(c).

The numerical values of  $f(a_0)$  and  $f(a_1)$  for both non-symmetric and symmetric cases are shown in Table 1. and Table 2. Since  $a_0$  and  $a_1$  are zeros of  $f$ , theoretically  $f(a_0) = 0$  and  $f(a_1) = 0$ . Figure 4 shows some Ahlfors maps of annulus shaped digital images using the proposed integral equation method.

**Table 1.** Numerical values of  $f(a_0)$  and  $f(a_1)$  with  $r = 0.4$

Non-Symmetric case $a_0 = 0.6, a_1 = \frac{-r}{a_0}$		
$n$	$f(a_0)$	$f(a_1)$
8	9.0(-02)	5.4(-02)
16	2.7(-03)	3.9(-03)
32	6.5(-06)	8.7(-06)
64	1.9220(-11)	2.4431(-11)
128	1.3271(-14)	8.2577(-15)
Symmetric case $a_0 = \sqrt{r}, a_1 = -\sqrt{r}$		
8	8.7(-02)	3.9(-02)
16	1.6(-03)	5.6122(-04)
32	9.0527(-07)	2.6040(-07)
64	3.6991(-13)	9.6021(-14)
128	3.9200(-15)	3.8750(-15)

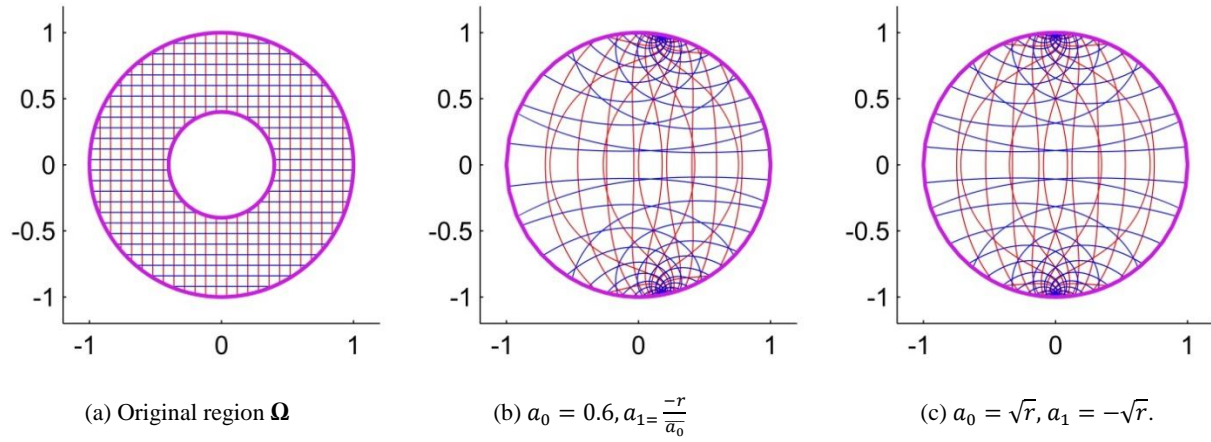
**Table 2.** Numerical values of  $f(a_0)$  and  $f(a_1)$  with  $r = 0.1$

Non-Symmetric case $a_0 = 0.6, a_1 = \frac{-r}{a_0}$		
$n$	$f(a_0)$	$f(a_1)$
8	5.82(-02)	6.01(-02)
16	1.4(-03)	1.4(-03)
32	4.9822(-07)	5.0099(-07)
64	4.4766(-14)	4.5830(-14)
128	3.5388(-15)	4.6887(-15)
Symmetric case $a_0 = \sqrt{r}, a_1 = -\sqrt{r}$		
8	2.0058(-04)	1.7918(-04)
16	1.8586(-08)	1.65839(-08)
32	6.2063(-16)	6.5799(-16)

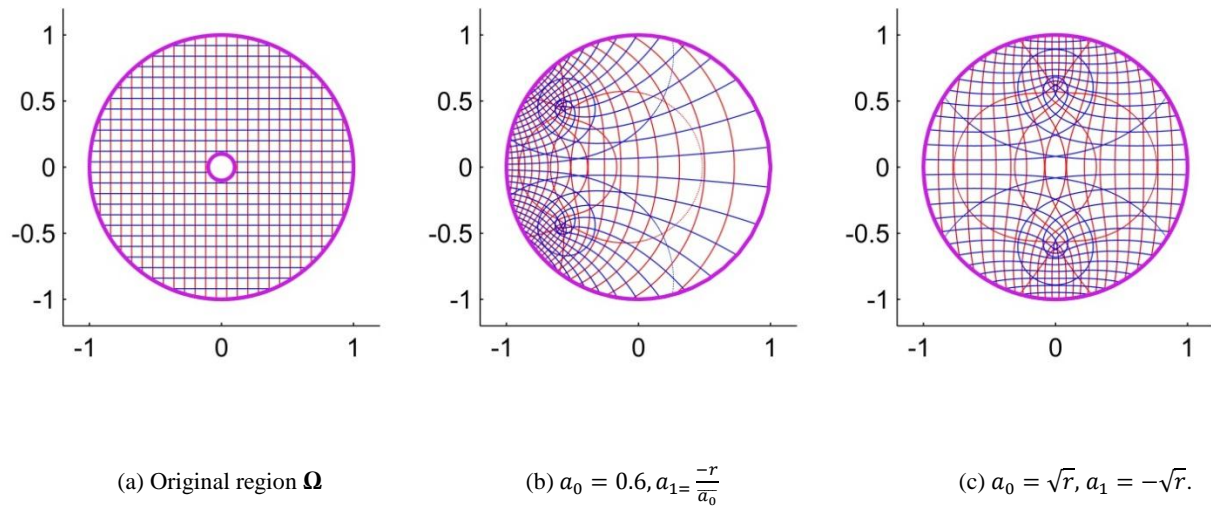
## 5.0 CONCLUSION

In this paper, we have constructed a boundary integral equation method for finding the Ahlfors map of multiply connected region onto a unit disk, provided that the zeros of Ahlfors map are known. We first derived an integral equation for boundary correspondence function  $\theta(t)$  of Ahlfors map and then used fast multipole method (FMM) for each iteration of the iterative method GMRES to solve the linear system obtained by discretization of our integral equation. Analytical method for computing the exact zeros of Ahlfors map for annulus region is presented in [21] and [2] but the problem of finding zeros for general doubly and higher connected regions remains unsolved. Probably the boundary integral equation used here and [20] can be combined with the approach used in [18] to obtain a method for computing zeros of Ahlfors map.

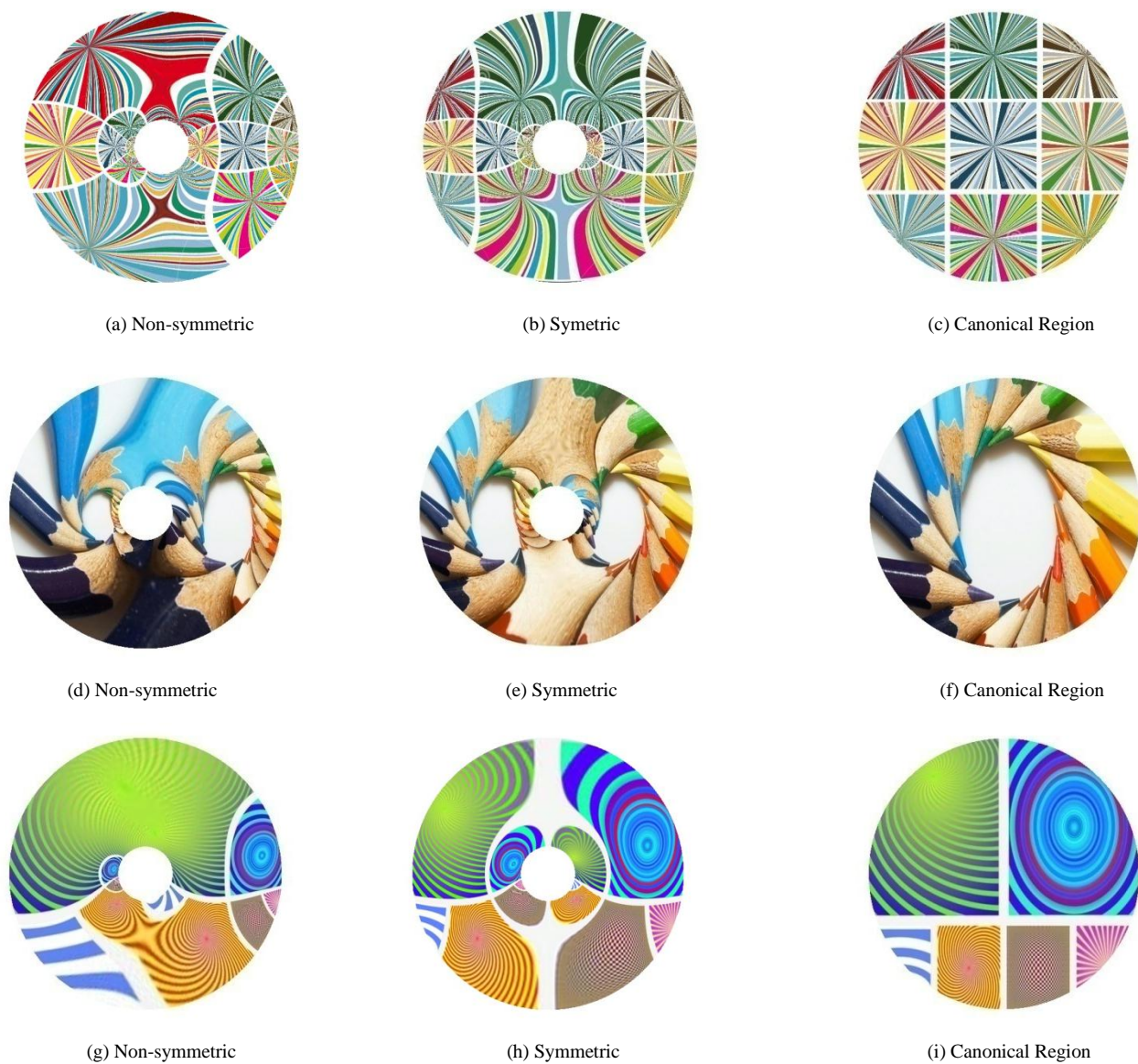




**Figure 2:** Numerical Ahlfors map of a region  $\Omega$  with  $r = 0.4$  for (b) non-symmetric case, and (c) symmetric case.



**Figure 3:** Numerical Ahlfors map of a region  $\Omega$  with  $r = 0.1$  for (b) non-symmetric case, and (c) symmetric case.



**Figure 4** Processed images under proposed numerical Ahlfors map with  $r = 0.1$  for non-symmetric cases  $a_0 = 0.6, a_1 = \frac{-r}{a_0}$ , and symmetric cases  $a_0 = \sqrt{r}, a_1 = -\sqrt{r}$ .



### Acknowledgment

This work was supported in part by the Malaysian Ministry of Education through the Research Management Centre (RMC), Universiti Teknologi Malaysia (FRGS Ref. No. R.J130000.7809.4F637). This support is gratefully acknowledged. The authors would like to thank the JT editor and the referee whose valuable comments and suggestions have led to this improved version of the manuscript.

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